Lecture No. 3

Symmetry and other Properties of Matrices and Operators

Matrices

$$\underline{\boldsymbol{A}}\,\underline{U} = \underline{P}$$

Pass $\underline{U}$ through $\underline{\boldsymbol{A}}$ to obtain $\underline{P}$

Operators

$$L(u) = p(x)$$

e.g.

$$L(u) = a_1 \frac{d^2u}{dx^2} + a_2 \frac{du}{dx} + a_3 u$$

Pass u through L to get $p(x)$

Symmetry of a Matrix

A matrix is symmetric if $\underline{\boldsymbol{A}} = \underline{\boldsymbol{A}}^{\boldsymbol{T}}$ $\left(a_{ij} = a_{ji}\right)$. Symmetry is a desirable matrix property. It saves on storage space and on the number of operations in manipulating and solving matrices.

Positive Definite Matrices

$$\underline{U}^T \underline{\boldsymbol{A}}\, \underline{U} = c$$

when $c > 0$ for all $\underline{U} \neq 0$, the matrix $\underline{A}$ is positive definite.

- We note that $\underline{U}^T \underline{U} > 0$ for all $\underline{U}$.
- A symmetrical positive definite matrix is desirable relative to *both* the actual implementation of its solution and the properties of the actual solution (since matrix eigenvalues are > 0).
- How can we extend the concepts of symmetry and positive definiteness to an operator?

Alternative method for establishing symmetry

Let's examine an alternative for establishing symmetric. Assume that the matrix $\underline{\boldsymbol{A}}$ is symmetric:

$$\underline{Y}^T \underline{\boldsymbol{A}}\, \underline{X} = \underline{Y}^T \underline{\boldsymbol{A}}^T \underline{X} \quad (\text{since } \underline{\boldsymbol{A}} = \underline{\boldsymbol{A}}^T)$$

$$\Rightarrow$$

$$\underline{Y}^T \underline{\boldsymbol{A}}\, \underline{X} = \underline{Y}^T \underline{C} \quad \text{where} \quad \underline{C} = \underline{\boldsymbol{A}}^T \underline{X}$$

$$\Rightarrow$$

$$\underline{Y}^T \underline{\boldsymbol{A}}\, \underline{X} = \underline{C}^T \underline{Y} \quad (\text{since } \underline{Y}^T \underline{C} \text{ and } \underline{C}^T \underline{Y} \text{ are scalar products})$$

$$\Rightarrow$$

$$\underline{Y}^T \underline{\boldsymbol{A}}\, \underline{X} = (\underline{\boldsymbol{A}}^T \underline{X})^T \underline{Y}$$

$$\Rightarrow$$

$$\underline{Y}^T \underline{\boldsymbol{A}}\, \underline{X} = \underline{X}^T \underline{\boldsymbol{A}}^{T^T} \underline{Y} \quad \text{since} \quad (\underline{\boldsymbol{M}}\, \underline{N})^T = \underline{N}\, \underline{\boldsymbol{M}}^T$$

$$\Rightarrow$$

$$\underline{Y}^T \underline{\boldsymbol{A}}\, \underline{X} = \underline{X}^T \underline{\boldsymbol{A}}\, \underline{Y} \quad \text{since} \quad \underline{\boldsymbol{A}}^{T^T} = \underline{\boldsymbol{A}}$$

Therefore when $\underline{\mathbf{A}}$ is symmetrical:

$$\underline{Y}^T \underline{\boldsymbol{A}}\, \underline{X} = \underline{X}^T \underline{\boldsymbol{A}}\, \underline{Y}, \text{ defines an inner product}$$

$$\Rightarrow$$

$$< \underline{Y}, \underline{\boldsymbol{A}}\, \underline{X} > = < \underline{X}, \underline{\boldsymbol{A}}\, \underline{Y} >, \text{ defines an inner product}$$

We now extend this definition to operators $L\,(u)$. In general we can state that

$$< v, L(u) > = < u,\ L^*(v) > \ + \int_{\Gamma} (F(v)G(u) - F(u)G^*(v))\, dS$$

Thus in general we can always exchange u and v in a manner similar to the interchange of vectors $\underline{X}$ and $\underline{Y}$. Only now we execute this interchange ***through an integration by parts procedure***.

- The operator L^* is the adjoint of L. If $L = L^*$ then L is self adjoint (in which case we also have $G = G^*$). Self adjointness of an operator is analogous to symmetry of a matrix.
- F and G are differential operators which fall out of the integration by parts procedure

  $F\,(u)$ defines the essential b.c.'s which must be enforced at some point in the domain for uniqueness

  $G\,(u)$ defines the natural b.c.'s

- Positive definiteness of a self adjoint operator is defined by the requirement that:

$$< L(u), u > > 0, \qquad \forall u \neq 0$$

  Which satisfy homogeneous b.c.'s.

- To establish positive definiteness, we look at the halfway point of the integration by parts of $< L(u), u >$.

<u>Example</u>

$$L(u) = \frac{d^2u}{dx^2} \qquad 0 < x < 1$$

$$< v, L(u) > = \int_0^1 v \frac{d^2u}{dx^2} dx$$

Integrating by parts once, defines the halfway point of the process in which we note equal derivatives on *u* and *v*.

$$\langle v, L(u) \rangle = \left[v \frac{du}{dx} \right]_0^1 - \int_0^1 \frac{dv}{dx} \frac{du}{dx} \, dx$$

Integrating by parts a second time, completes the transformation:

$$\langle v, L(u)\rangle = \left[v\frac{du}{dx} - u\frac{dv}{dx}\right]_0^1 + \int_0^1 u\frac{d^2v}{dx^2}\ dx$$

Comparing this to our “generic” transformation (or integration) defines the various operators as well as boundary conditions:

$$L(u) = \frac{d^2u}{dx^2}$$

$$L^*(v) = \frac{d^2v}{dx^2}$$

Therefore $L = L^*$ indicates that the operator is self adjoint (symmetric). The boundary conditions are established as:

first term	second term	
$F(v) = v$	$F(u) = u$	u represents the essential b.c
$G(u) = \frac{du}{dx}$	$G^*(v) = \frac{dv}{dx}$	$\frac{du}{dx}$ represents the natural b.c

Assuming $v = u$ and homogeneous b.c.'s the first integration by parts yields:

$$\int_0^1 uL(u)dx = -\int_0^1 \left(\frac{du}{dx}\right)^2 dx$$

Therefore $L(u)$ is negative definite.

Summary of Symmetry and other properties of Matrices and Operators

- The matrix $\underline{\boldsymbol{A}}$ is symmetrical if for any vectors $\underline{X}$ and $\underline{Y}$

$$\underline{X}^T\left(\underline{\boldsymbol{A}}\,\underline{Y}\right) = \underline{Y}^T\left(\underline{\boldsymbol{A}}\,\underline{X}\right) \;\rightarrow\; <\underline{X}, \underline{\boldsymbol{A}}\,\underline{Y}> = <\underline{Y}, \underline{\boldsymbol{A}}\,\underline{X}>$$

- We extend this inner product definition of symmetry to operators: We can *always* state

$$< v, L(u) > = < u,\ L^*(v) > + \int_\Gamma (F(v)\, G(u) - F(u)G^*(v))dS$$

Thus we interchange the roles of u and v through an integration by parts procedure. We also define b.c.'s through this process!

Important Consequences of the Integration by Parts Procedure

1. Provides information about the operator L

   If $L = L^*$, the operator is self adjoint (symmetrical).

   The procedure also allows us to establish whether the operator is positive definite (we look at the halfway point).

2. The procedure allows us to determine the characteristics of the matrix produced for the Galerkin method and a given operator L.

3a. Defines the b.c.'s for the given operator.

essential b.c.'s $= F(u)$

natural b.c.'s $= G(u)$ and $G^*(u)$

3b. The procedure will also provide a mechanism for us to develop a "weak" formulation where we can relax *some* of the admissibility requirements for boundary condition satisfaction.

4. The procedure provides a mechanism for changing derivatives on operators.

We noted that derivatives taken at the halfway points of our integration by parts procedure were always lower. This will allow us to establish “weak” formulations where we can lower some of the admissibility for functional continuity.

Example

Consider $$L(u) = \frac{d}{dx}\left(a_o(x)\frac{du}{dx}\right) + a_1(x)\frac{du}{dx} + a_2(x)u$$

The first term represents diffusion, the second term convection and the third term decay.

Let's integrate by parts such that $\langle v, L(u)\rangle \rightarrow \langle u, L^*(v)\rangle$

Thus we define the inner product as:

$$< v, L(u) > = \int_V \left\{\frac{d}{dx}\left(a_o\frac{du}{dx}\right) + a_1\frac{du}{dx} + a_2 u\right\} v dx$$

Integrating the first term by parts we have:

$$\int_V \frac{d}{dx}\left(a_o\frac{du}{dx}\right) v dx = \int_V (v) d\left(a_o\frac{du}{dx}\right)$$

$$= \left| v a_o \frac{du}{dx}\right|_\Gamma - \int_V \; a_o\frac{du}{dx} d(v)$$

$$= \left| v a_o \frac{du}{dx}\right|_\Gamma - \int_V \; a_o\frac{du}{dx}\frac{dv}{dx} dx$$

Substituting into the definition of our inner product

$$< v, L(u) > = \left| v a_o \frac{du}{dx} \right|_{\Gamma} + \left\{ \int_V -a_o \frac{dv}{dx}\frac{du}{dx} + a_1 v \frac{du}{dx} + a_2 uv \right\} dx$$

Now perform a second integration by parts on the first term:

$$\int_V -a_o \frac{dv}{dx}\frac{du}{dx} dx = \int_V -a_o \frac{dv}{dx} du$$

$$= \left| -a_o \frac{dv}{dx} u \right|_{\Gamma} + \int_V u \, d\left(a_o \frac{dv}{dx} \right)$$

$$= \left| -a_o \frac{dv}{dx} u \right|_{\Gamma} + \int_V u \frac{d}{dx}\left(a_o \frac{dv}{dx} \right) dx$$

In addition perform an integration by parts on the second term:

$$\int_V a_1 v \frac{du}{dx} dx = \int_V a_1 v du = |a_1 v u|_\Gamma - \int_V u d(a_1 v) = |a_1 v u|_\Gamma - \int_V u \frac{d(a_1 v)}{dx} dx$$

Substituting into our halfway point of the integration by parts process, we complete the transformation:

$$< v, L(u) > = \left| a_o \frac{du}{dx} v + u\left(a_1 v - a_o \frac{dv}{dx}\right)\right|_\Gamma + \int_V \left\{\frac{d}{dx}\left(a_o \frac{dv}{dx}\right) - \frac{d(a_1 v)}{dx} + a_2 v\right\} u dx$$

Thus:

$$L(u) = \frac{d}{dx}\left(a_o(x) \frac{du}{dx}\right) + a_1(x) \frac{du}{dx} + a_2(x) u$$

$$L^*(v) = \frac{d}{dx}\left(a_o(x) \frac{dv}{dx}\right) - \frac{d}{dx}(a_1(x) v) + a_2(x) v$$

Therefore the operator L is <u>not</u> in general self adjoint.

- However, the diffusion and decay terms give symmetry and thus self adjointness. The convection term gives skew symmetry and is not self adjoint. Therefore if $a_1 \neq 0 \Rightarrow L$ is not self adjoint.

- Consider $a_1 = 0$. $L = L^*$ $\Rightarrow$ L is self adjoint

  Let's now establish the b.c.'s for this special case:

$$\int_V \left\{\frac{d}{dx}\left(a_o \frac{du}{dx}\right) + a_2 u\right\} v dx = \left| a_o \frac{du}{dx} v - u a_o \frac{dv}{dx} \right|_\Gamma + \int_V \left\{\frac{d}{dx}\left(a_o \frac{dv}{dx}\right) + a_2 v\right\} u dx$$

Since this is a self adjoint case $L = L*$ and $G = G*$

$F(v) \rightarrow$ <u>essential</u> b.c.'s $F(v) \equiv v \rightarrow$ <u>must</u> prescribe the function (for uniqueness)

$G(u) \rightarrow$ <u>natural</u> b.c.'s $G(u) = a_o \frac{du}{dx} \rightarrow$ <u>can</u> prescribe derivatives of the function

Prescription of b.c.’s

- Essential b.c.’s must be specified at least one point on the boundary.

- We cannot specify natural and essential b.c.’s at the same point.

  Examine the following simple case:

$$\frac{d^2w}{dx^2} = f \quad \text{where} \quad f = \text{constant}$$

  Integrating we have:

$$w = c_o + c_1 x + \frac{1}{2} f x^2$$

  Where c_o and c_1 are the constants of integration

However $\frac{dw}{dx} = c_1 + fx \quad \rightarrow$ the c_o term disappears

Therefore if the b.c.’s only involve $\frac{dw}{dx}$, we can only compute one coefficient!

Therefore specifying the function w ***as a b.c. is “essential” to getting a unique solution.***

Halfway point of the integration by parts

When the operator is symmetrical, you can select b.c.'s at the halfway point of the integration by parts procedure. Therefore the b.c.'s picked up during the second half of the integration by parts are the same (except that *u* and *v* are interchanged and with a minus sign).

At the halfway point:

$$\int_{v} \left\{ \frac{d}{dx}\left(a_o \frac{du}{dx}\right) + a_2 u \right\} v dx = \left| a_o \frac{du}{dx} v \right|_{\Gamma} + \int_{V} \left\{ -a_o \frac{dv}{dx}\frac{du}{dx} + a_2 uv \right\} dx$$

Defines

$$G(u) = a_o \frac{du}{dx}$$

$$F(v) = v$$

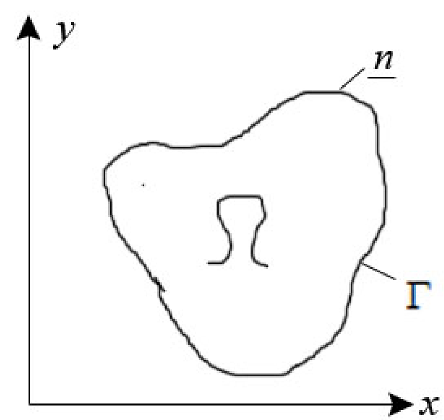
Also note that we have equal order derivatives on *u* and *v* at this point of the integration!

Integration by parts for 2-D

Use Green's theorem:

$$\int\int_{\Omega} f g_{,x}\, dxdy = \int_{\Gamma} f\ g\ \cos(n,x) d\Gamma - \int\int_{\Omega} g\ f_{,x}\, dxdy$$

Where cos (n, x) defines the direction cosine.



Example

Consider Poisson's equation:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + f(x,y) = 0$$

$$\Rightarrow$$

$$\nabla^2 u + f(x,y) = 0$$

We define the inner product as:

$$\int \int_{\Omega} (u_{,xx} + u_{,yy}) v dx dy = \int_{\Gamma} (u_{,x}\alpha_{nx} + u_{,y}\alpha_{ny}) v d\Gamma$$

$$- \int \int_{\Omega} (u_{,x}v_{,x} + u_{,y}v_{,y}) d\Omega$$

essential b.c. is $u = \overline{u}$ (overbar indicates user specified)

natural b.c. is $\frac{\partial u}{\partial n} = u_{,x}\alpha_{nx} + u_{,y}\alpha_{ny} = \overline{q}$

Integrating once more we conclude that the operator *L* is self adjoint.